Simple Harmonic Motion

- Periodic Motion — Repeats itself after fix. time interval.

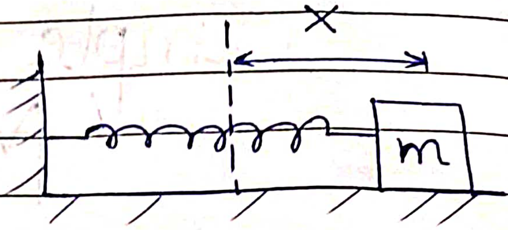
- Oscillatory Motion — To & fro motion.

- Oscillatory motion is NOT necessarily periodic motion.

- S.H.M. — Special case of Oscillatory motion. Here, $\boxed{\vec{F} \propto (-\vec{x})}$

where $\vec{x}$ is disp. from mean posi.

Eq^n of S.H.M



$$\vec{F} = (-k)\vec{x}$$

$$\Rightarrow m\vec{a} = (-k)\vec{x}$$

$$\Rightarrow a = \left(\frac{-k}{m}\right)x = v\left(\frac{dv}{dx}\right)$$

$$\Rightarrow \int v\,dv = \int\left(\frac{-k}{m}\right)x\,dx$$

$$\Rightarrow \left(\frac{v^2}{2}\right) = \left(\frac{-k}{m}\right)\left(\frac{x^2}{2}\right) + C$$

Now, at $x = A$ (extreme posᵗ), $v = 0$.

$$\Rightarrow \left(\frac{v^2}{2}\right) = \left(\frac{-k}{m}\right)\left(\frac{x^2}{2}\right) + \left(\frac{k}{m}\right)\left(\frac{A^2}{2}\right)$$

$$\Rightarrow v^2 = (k/m)(A^2 - x^2)$$

$$\Rightarrow \boxed{v = \omega\sqrt{A^2 - x^2}} \quad \text{where } \omega = \sqrt{\frac{k}{m}}$$

$$\Rightarrow \int \frac{dx}{\sqrt{A^2 - x^2}} = \int \omega\,dt$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{A}\right) = \omega t + C$$

At $t=0$, assume $x = 0$.

$\Rightarrow \quad \sin^{-1}(x/A) = \omega t$

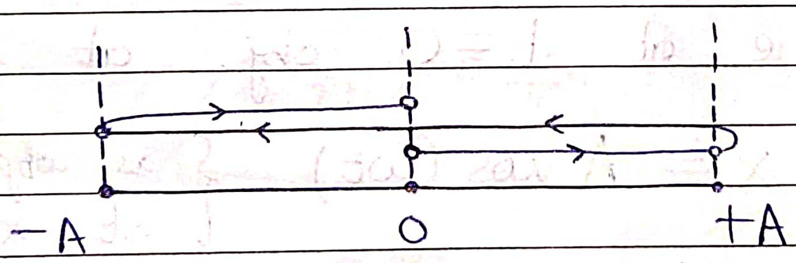
$\Rightarrow \quad \boxed{x = A\sin(\omega t)}$ where $\omega = \sqrt{\frac{k}{m}}$.

Now, A = Amplitude i.e. max disp. from mean posⁿ.

& ω = Angular freq.

& T = Time Period i.e. time after which motion repeats.

$\Rightarrow \quad \boxed{T = 2\pi/\omega}$



Time to go from O to $+A$ = $T/4$
" " " " $+A$ to O = $T/4$
" " " " O to $-A$ = $T/4$
" " " " $-A$ to O = $T/4$

Q) Find first time to reach $+A/2$ from O.

A) $x = A \sin\omega t \Rightarrow \left(\frac{A}{2}\right) = A \sin\omega t$

$\Rightarrow \sin\omega t = \left(\frac{1}{2}\right) \Rightarrow t = \left(\frac{\pi}{6\omega}\right) \Rightarrow \boxed{t = \left(\frac{T}{12}\right)}$

Q) Find time to reach $+A$ from $+A/2$.

A) Method 1: $t_{req.} = T/4 - T/12$ — from above Q

$\Rightarrow t = T/6$

Method 2: Since $t_{+\frac{A}{2} \to +A} = t_{+A \to +\frac{A}{2}}$,

we assume at $t=0$ obj. at $x = +A$.

Now, $x = A \cos(\omega t)$ {as obj. init at $x = +A$}

$\Rightarrow \left(\frac{A}{2}\right) = A \cos\omega t$

$\Rightarrow t = \left(\frac{\pi}{3\omega}\right) \Rightarrow \boxed{t = T/6}$

★ 'x' is **ALWAYS** measured from Mean Posi.

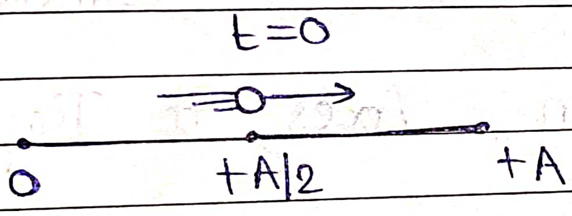
General Eq^n of SHM

$$\boxed{x = A\sin(\omega t + \phi)}$$

where ϕ = Phase Angle. $\phi \in \cancel{[0, 2\pi]}$ $[-\pi, +\pi]$

It tells us about init. posⁿ of obj.

Q) find phase angle.



A) $x = A\sin(\omega t+\phi) \Rightarrow (A/2) = A\sin\phi \Rightarrow \sin\phi = 1/2$

$$\Rightarrow \phi = \pi/6,\ 5\pi/6$$

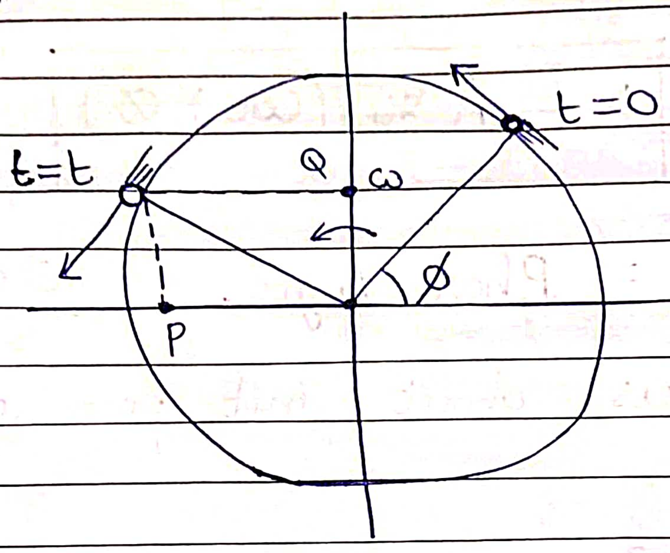
Now, $v = A\omega\cos(\omega t+\phi) \Rightarrow v_0 = A\omega\cos\phi$

for $v_0 > 0$, $\phi = \pi/6$

for $v_0 < 0$, $\phi = 5\pi/6$

$$\boxed{\phi = \pi/6}$$

Comparison with Uniform Circular Motion



Proj. of Obj. on Axes, Diameters & Tangents performs SHM.

$$P \equiv x_P = R\cos(\omega t + \phi)$$

$$Q \equiv x_Q = R\sin(\omega t + \phi)$$

Velocity & Acceleration

Velocity — $x = A\,s_{\omega t}$

$$\Rightarrow \boxed{v = A\omega\, c_{\omega t}}$$

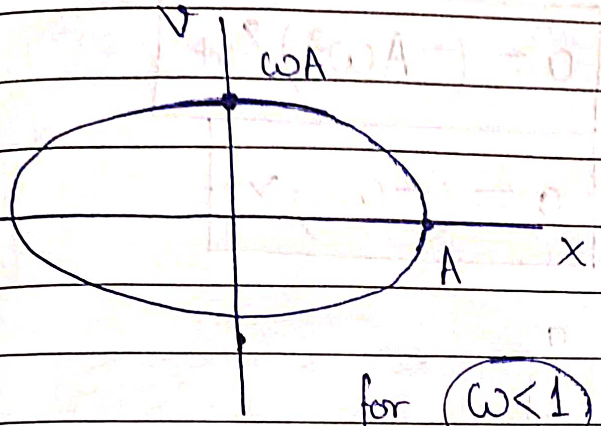
$$\Rightarrow \boxed{v = \omega\sqrt{A^2 - x^2}}$$

$$\Rightarrow v_{max.} = \omega A \quad \text{at} \quad x = 0$$

Now, $v = \omega\sqrt{A^2 - x^2} \Rightarrow \left(\frac{v}{\omega A}\right)^2 + \left(\frac{x}{A}\right)^2 = 1$

$\Downarrow$

Ellipse !



for $\omega < 1$

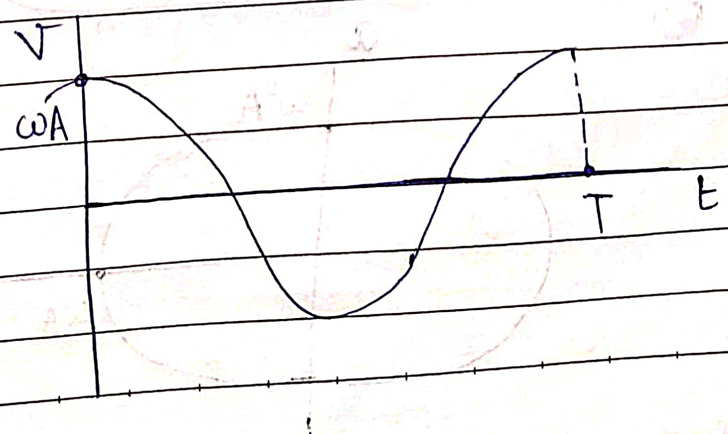
Q) Vel. of particle at $x = 6$ is 8.
" " " " $x = 8$ is 6.
Find time period.

A) $v_1^2 = (\omega^2)(A^2 - x_1^2)$
$v_2^2 = (\omega^2)(A^2 - x_2^2)$

$(v_2^2 - v_1^2) = (\omega^2)(x_1^2 - x_2^2)$

$\Rightarrow \omega = 1$

$\Rightarrow \boxed{T = 2\pi}$

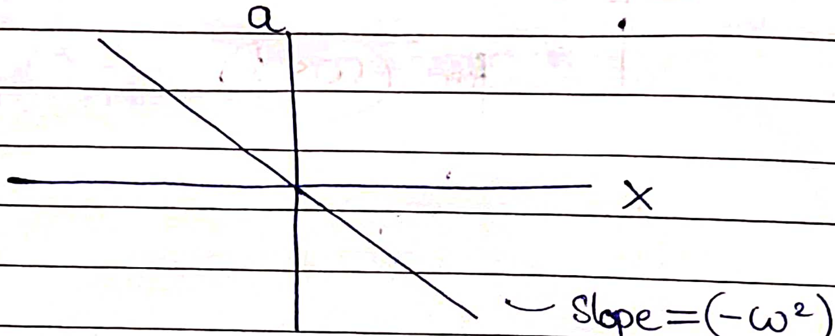


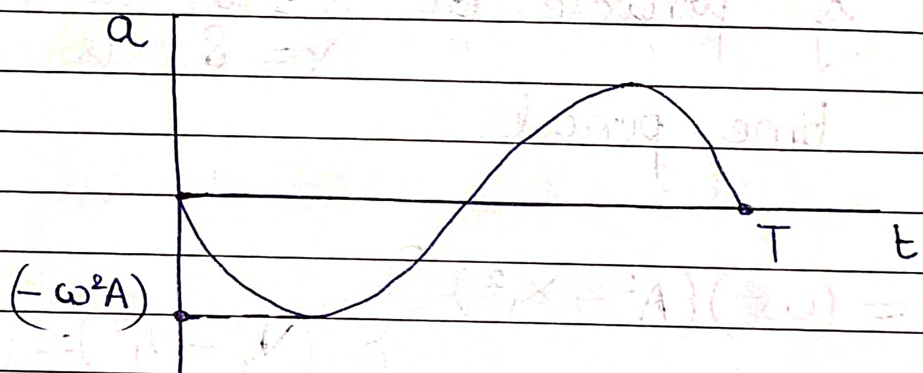
Date: ______ Page: ______

Acceleration — $v = A\omega \cos\omega t$

$\Rightarrow \boxed{a = (-A\omega^2)\sin\omega t}$

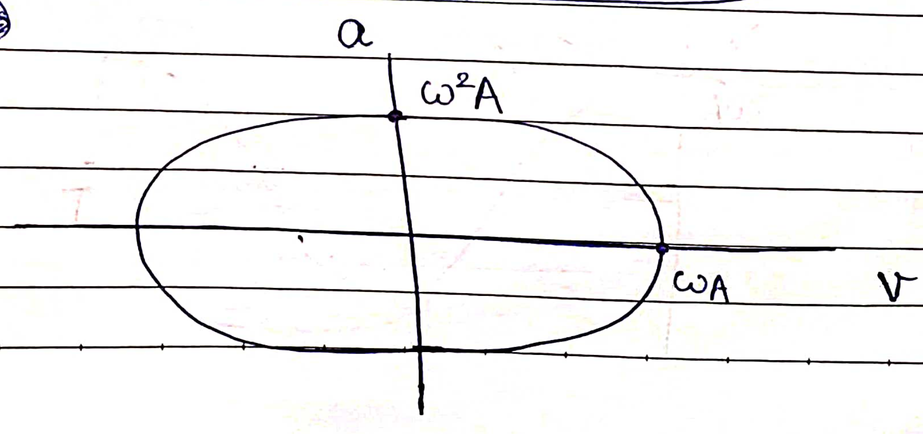
$\Rightarrow \boxed{a = (-\omega^2)x}$

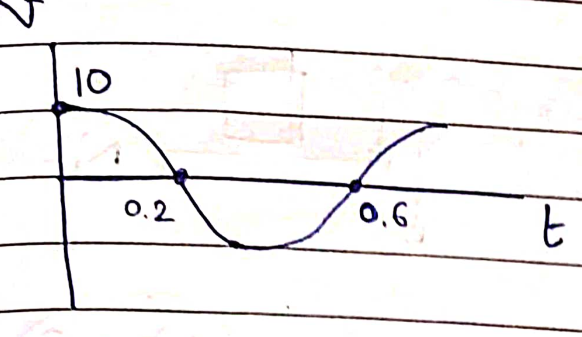




Now, $v/\omega = A\cos\omega t$, $(-a)/\omega^2 = A\sin\omega t$

$\Rightarrow (v/\omega A)^2 + (a/\omega^2 A)^2 = 1 \Rightarrow$ Ellipse !



Q) 

find eq^n of SHM.

A) $v = A\omega \cos\omega t \Rightarrow \boxed{10 = A\omega}$

$\cancel{\Rightarrow 0 = A\omega \cos\omega(0.2) \Rightarrow}$

Now, $T/2 = (0.6 - 0.2) \Rightarrow T = (0.8)$

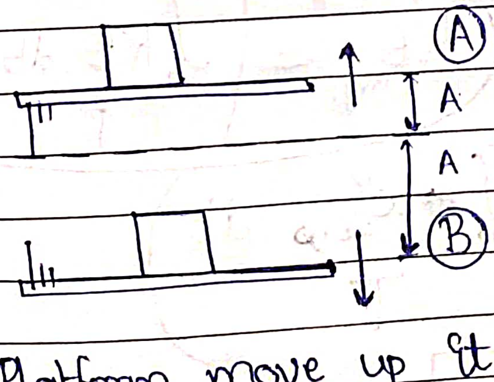
$\cancel{\Rightarrow \omega = \left(\frac{5}{\pi}\right)}$

Multiplying, $(0.8)(10) = A\omega T = 2\pi A = 8$

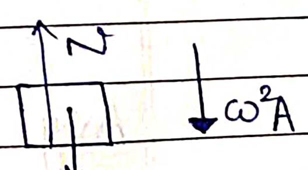
$\Rightarrow A = \left(\frac{4}{\pi}\right)$

Also, $\omega = \left(\frac{10}{A}\right) \Rightarrow \omega = \left(\frac{5}{2}\right)\pi$

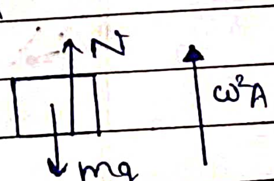
$$\Rightarrow \boxed{x = \left(\frac{4}{\pi}\right) \sin\left(\frac{5\pi t}{2}\right)}$$

Q) 

Platform move up & down.
Where obj. leave?

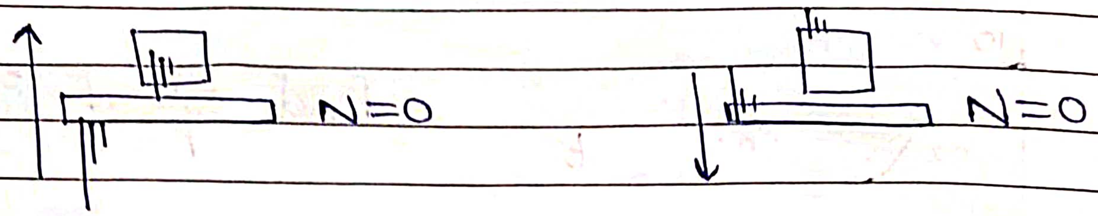
A) At (A), 

$N = mg - m\omega^2 A$

At (B), 

$N = mg + m\omega^2 A$

More likely to leave at (A).



Obj. will leave platform when $N=0$

It doesn't matter if platform moving up. or down.

Energy

Kinetic Energy — $KE = \frac{1}{2}mv^2$

$$\Rightarrow \boxed{KE = \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\phi)}$$

$$\Rightarrow \boxed{KE = \frac{1}{2}m\omega^2(A^2-x^2)}$$

Avg. K.E. (wrt time): $KE_{avg.} = \left(\frac{\int_0^T KE\,dt}{T}\right)$

$$\Rightarrow KE_{avg.} = \frac{1}{2}m\omega^2A^2\left(\frac{\int_0^T \cos^2_{\omega t+\phi}\,dt}{T}\right)$$

$$\Rightarrow \boxed{KE_{avg.} = \frac{1}{4}m\omega^2A^2}$$

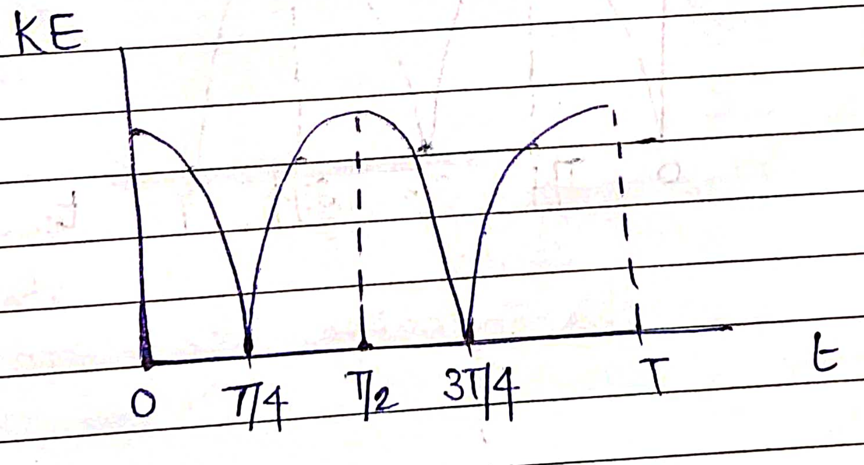
$$\overline{\left(\sin^2(\theta)\right)} = \left(\frac{\int_0^{2\pi} \sin^2(\theta)\, d\theta}{2\pi}\right) = \frac{1}{2}$$

$$\overline{\left(\cos^2(\theta)\right)} = \left(\frac{\int_0^{2\pi} \cos^2(\theta)\, d\theta}{2\pi}\right) = \frac{1}{2}$$

Avg. K.E. (wrt. disp.) : $KE_{avg.} = \left(\frac{\int_0^A KE\, dx}{A}\right)$

$\Rightarrow$ ~~$KE_{avg.} = \frac{1}{2} m\omega^2$~~

$$\Rightarrow \boxed{KE_{avg.} = \frac{1}{3} m\omega^2 A^2}$$



Date: ______ Page: ______

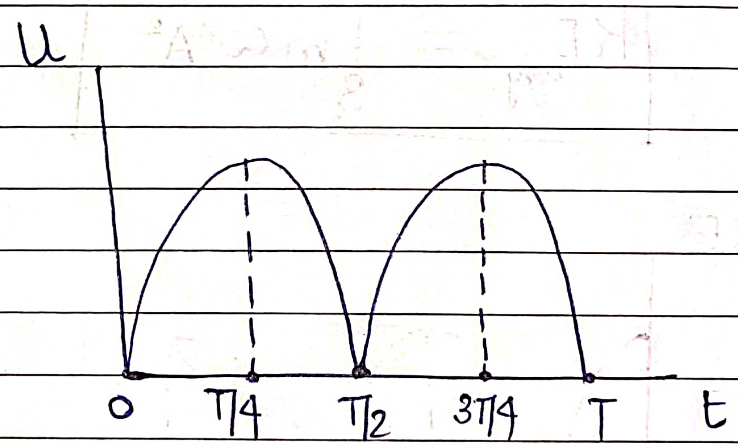
Potential Energy — $U = \frac{1}{2} K x^2$

$\Rightarrow \boxed{U = \frac{1}{2} m\omega^2 x^2}$

$\Rightarrow \boxed{U = \frac{1}{2} m\omega^2 A^2 \sin^2_{(\omega t+\varphi)}}$

Avg. U (wrt time) : $\boxed{U_{avg.} = \frac{1}{4} m\omega^2 A^2}$

- **Avg. U (wrt disp)** : $\boxed{U_{avg.} = \frac{1}{6} m\omega^2 A^2}$

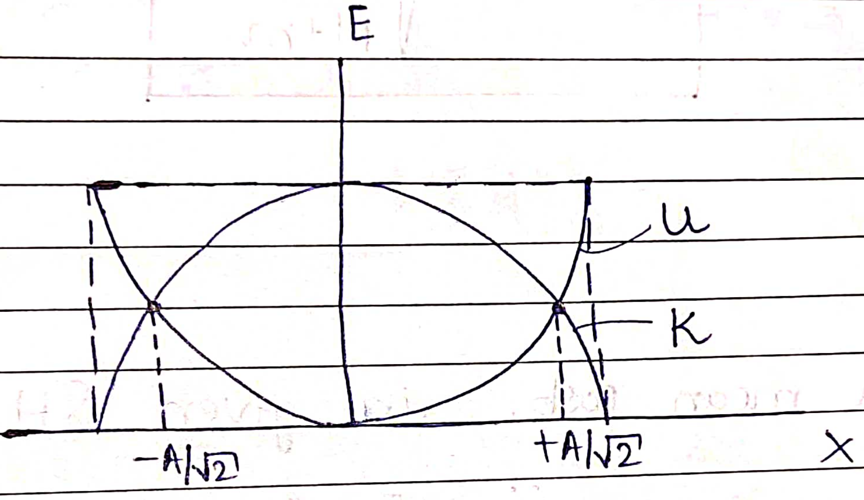


Total Energy — T.E. = K.E. + P.E.

$\Rightarrow E = K + U \Rightarrow \boxed{E = \frac{1}{2} m\omega^2 A^2}$

$$\Rightarrow E = \frac{1}{2} m \left(\sqrt{\frac{K}{m}}\right)^2 A^2 \Rightarrow E = \frac{1}{2} kA^2$$

⇒ Total Energy does NOT depend on Mass !



$K = U$ at $|X| = A/\sqrt{2}$

Q) A mass M moving in SHM with amplitude A. A mass m is put on it at mean posⁿ. Both move together. Find new amplitude.

A) (Loss in Energy) = (Work done by friction) = $\Delta K = \Delta K_{wrt\ CoM}$

(as V_{CoM} NOT change in small time)

$$\Rightarrow W_f = \left(\frac{-1}{2}\right)\left(\frac{Mm}{M+m}\right)(\omega A)^2$$

$$E' = E + W_f = \frac{1}{2}M(\omega A)^2 - \frac{1}{2}\left(\frac{Mm}{M+m}\right)(\omega A)^2$$

$$\Rightarrow \frac{1}{2}(M+m)\,\omega^2 (A')^2 = \frac{1}{2}(M+m)\,\omega^2 A^2 \left(\frac{M}{M+m}\right)^2$$

$$\Rightarrow \boxed{A' = A\sqrt{\frac{M}{M+m}}}$$

Q) find mean posⁿ. in given SHM eqⁿ.

$$\ddot{x} = (5 - 4x)$$

A) $\ddot{x} = (-4)(x - 5/4)$. Let $X = (x - 5/4)$

$\Rightarrow \boxed{\ddot{X} = (-4)X}$ $\Rightarrow \ddot{X} = \ddot{x}$

At mean posⁿ, $X = 0 \Rightarrow x = 5/4$

Particle do SHM about $x = 5/4$.

Q) find mean post. in given SHM eq^n

$$U = (ax^2 + bx + c)$$

A) $F = \left(-\dfrac{dU}{dx}\right) \Rightarrow F = -(2ax + b)$

$$\Rightarrow \ddot{x} = \left(\frac{-2a}{m}\right)\left(x + \frac{b}{2a}\right)$$

$\Rightarrow$ SHM abt. $\boxed{x = \left(\dfrac{-b}{2a}\right)}$

Q) Pt. $U = a(1 - \cos bx)$ represent eq^n of SHM for small value of x.

A) $F = \left(-\dfrac{dU}{dx}\right) = (-a)\sin bx \Rightarrow \boxed{F = (-ab)x}$

Q) find no. of SHMs in $y = A\sin^2(t/2)\cos 10t$

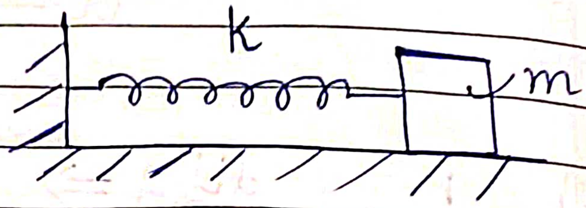
A) $y = \left(\dfrac{A}{2}\right)(1 - \cos t)\cos 10t = \left(\dfrac{A}{2}\right)\cos 10t - \left(\dfrac{A}{4}\right)(2\cos t \cos 10t)$

$$\Rightarrow \boxed{y = \left(\frac{-A}{4}\right)\cos 9t + \left(\frac{A}{2}\right)\cos 10t + \left(\frac{-A}{4}\right)\cos 11t}$$

$\Rightarrow$ Combination of 3 SHM

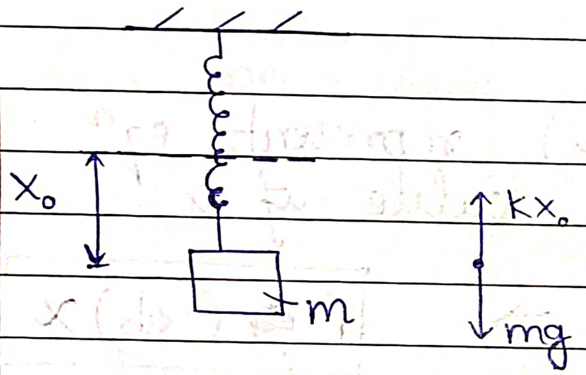
Examples of SHM

1) Horizontal Spring –



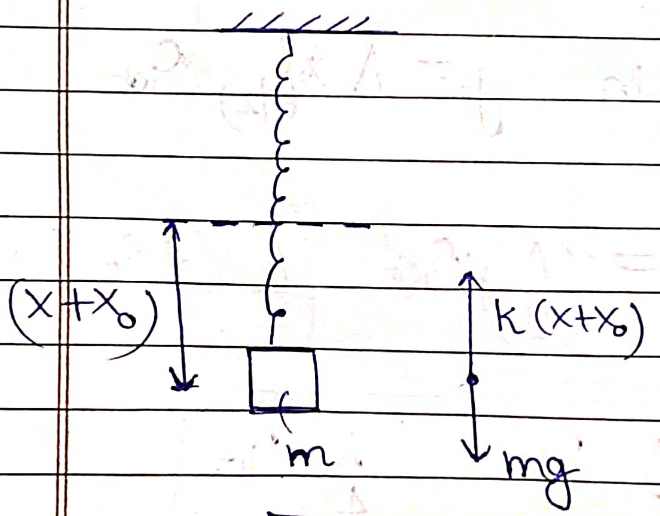
$$\boxed{T = 2\pi \cancel{\sqrt{\frac{k}{m}}}}\left(\sqrt{m/k}\right)$$

2) Vertical Spring –



$$kx_0 = mg$$

$$\Rightarrow \cancel{\frac{k}{m}}\left(\frac{m}{k}\right) = \left(\frac{x_0}{g}\right)$$



$$F_{restoring} = k(x+x_0) - mg$$

$$\Rightarrow F = kx$$

$$\Rightarrow \vec{F} = (-k)\vec{x}$$

$$\Downarrow$$

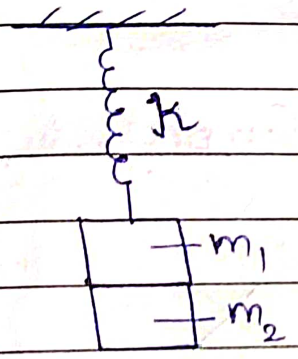
$$\boxed{T = 2\pi\sqrt{\frac{x_0}{g}}} \Leftarrow T = 2\pi \cancel{\frac{k}{m}} \sqrt{\frac{m}{k}}$$

Here, T does <u>NOT</u> depend on 'g' as

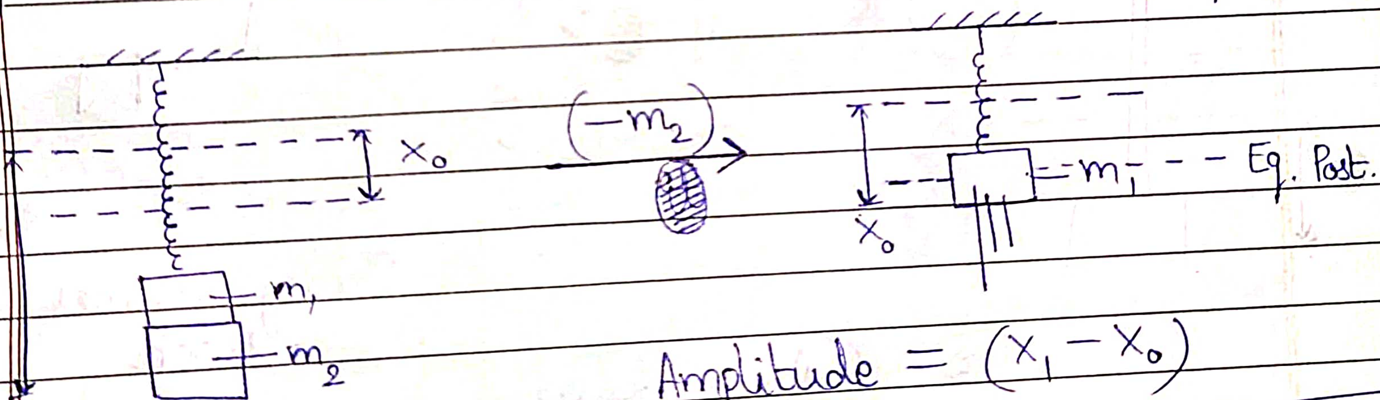
$$\left(m/K\right) = \left(x_0/g\right) = \text{Const.}$$

Q) Objs at eq.

m_2 suddenly removed. Find amplitude of m_1's SHM.



A) 'm_1' will do SHM about its eq. pt.

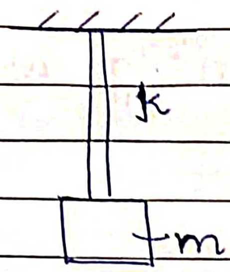


Amplitude $= (x_1 - x_0)$

$$= \frac{(m_1+m_2)g}{k} - \frac{m_1 g}{k} \Rightarrow \boxed{A = \left(\frac{m_2 g}{k}\right)}$$

Date: ____ Page: ____

Q) ☆

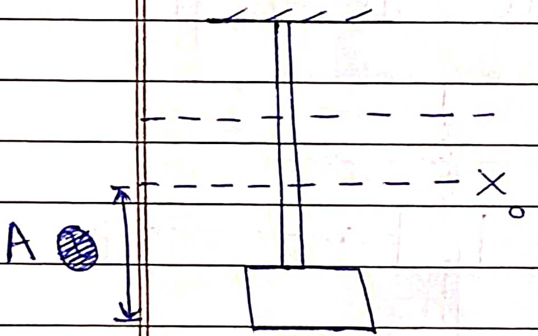


Extensible string with ~~string factor~~ force const. k.

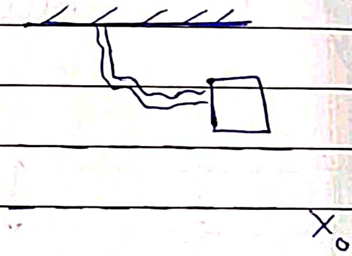
Find max. amplitude of SHM that can be performed by 'm'.

A)

	Stretch	Compress
Spring	✓	✓
String	✓	✗



if $A > x_0$ $\longrightarrow$



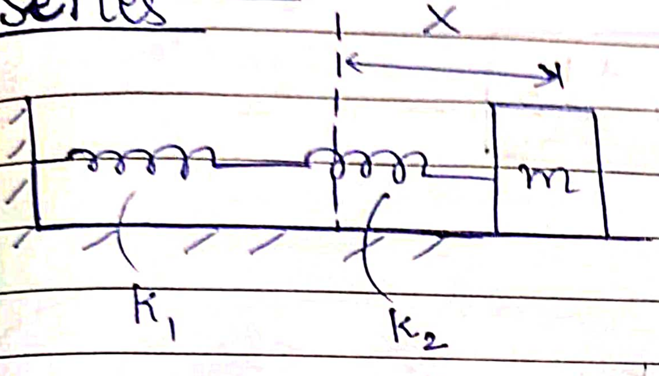
String becomes slack

$\Rightarrow$ Restoring force vanishes

$\Rightarrow$ SHM stops.

Combination of Springs

1) Series —



Let spring 1 stretch by 'x_1' & spring 2 stretch by 'x_2'

Now, $x = x_1 + x_2$

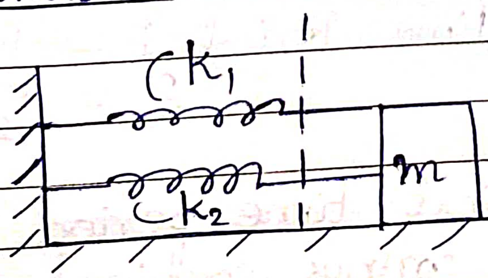
Since springs massless, force is same throughout.

$\Rightarrow \quad x = F/k_{eq}, \quad x_1 = F/k_1, \quad x_2 = F/k_2$

So $x = x_1 + x_2 \Rightarrow F/k_{eq} = F/k_1 + F/k_2$

$$\Rightarrow \boxed{\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}}$$

2) Parallel —

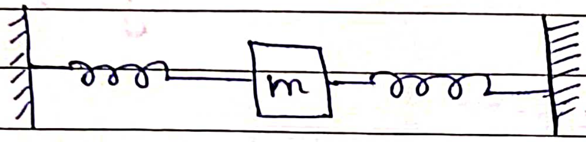


Both springs stretch by 'x'.

Now, $F = F_1 + F_2$

$$\Rightarrow \quad K_{eq}\,x = k_1 x + k_2 x$$

$$\Rightarrow \quad \boxed{K_{eq} = k_1 + k_2}$$

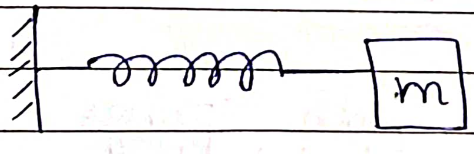
★ 

This is also || combination.

★ In identifying type of combination,

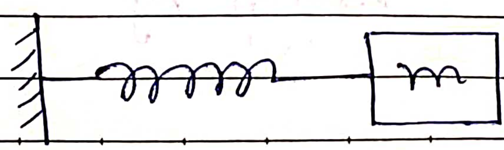
(Parallel) $\Leftrightarrow$ (Ext^n or compression in all springs same & = disp. of body)

(Series) $\Leftrightarrow$ (Sum of ext^n & compression of all springs = disp. of body.)

Q) 

Time Period 1 = T_1

" " 2 = T_2

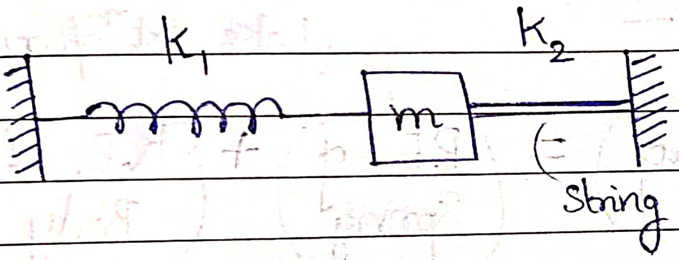


Find time period if springs combined in series.

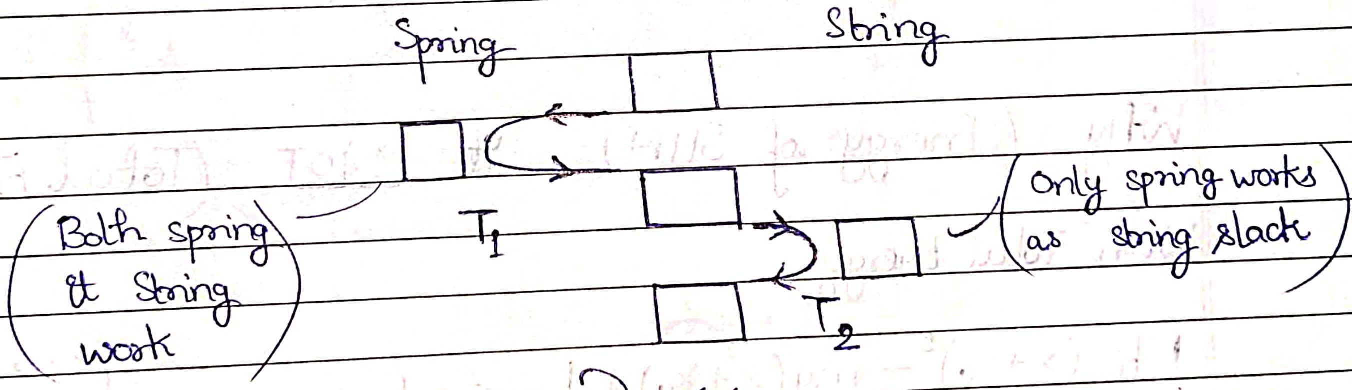
A) $T_1 = \frac{2\pi\sqrt{m}}{\sqrt{k_1}}$, $\frac{2\pi\sqrt{m}}{\sqrt{k_2}} = T_2$

$$T = \frac{2\pi\sqrt{m}}{\sqrt{k}} \Rightarrow T^2 = (4\pi^2 m)\left(\frac{1}{k}\right) = (4\pi^2 m)\left(\frac{1}{k_1} + \frac{1}{k_2}\right)$$

$$\Rightarrow \boxed{T^2 = T_1^2 + T_2^2}$$

Q) 

Find time period.

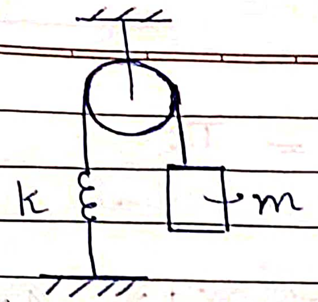
A) 

$$T_1 = \frac{\pi\sqrt{m}}{\sqrt{k}} = \frac{(\pi\sqrt{m})}{\sqrt{k_1+k_2}} \cancel{\frac{\sqrt{k_1k_2}}{\sqrt{k_1+k_2}}}$$

$$T_2 = \frac{\pi\sqrt{m}}{\sqrt{k_1}}$$

$$\Rightarrow \boxed{T = (\pi\sqrt{m})\left(\frac{1}{\sqrt{k_1+k_2}} + \frac{1}{\sqrt{k_1}}\right)}$$

358



Q) find time period

A) Restoring force $= kx \Rightarrow$ $\boxed{T = 2\pi \frac{\sqrt{m}}{\sqrt{k}}}$

Energy Method —

(take ext^n after mean posit.)

Concept : $\begin{pmatrix}\text{Energy of}\\ \text{SHM}\end{pmatrix} = \begin{pmatrix}\text{P.E. of}\\ \text{Spring}\end{pmatrix} + \begin{pmatrix}\text{K.E. of}\\ \text{Body}\end{pmatrix}$

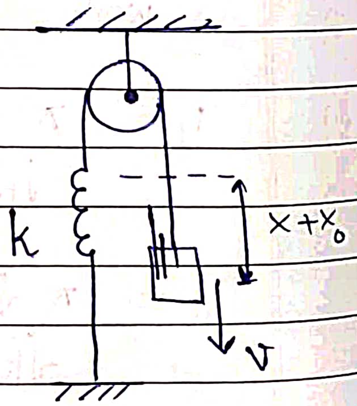
is Const. always.

Why (Energy of SHM) & NOT (Total Energy)?

With Total Energy,

$$\frac{1}{2}k(x+x_0)^2 - mg(x+x_0) + \frac{1}{2}mv^2 = \text{Const.}$$

where $kx_0 = mg$



$\left(\frac{d}{dx}\right)$

$$k(x+x_0) - mg + mv\left(\frac{dv}{dx}\right) = 0$$

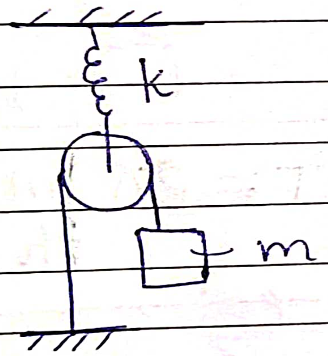
$$\Rightarrow kx + m\left(\frac{dv}{dt}\right) = \underbrace{mg - kx_0}_{=0}$$

(Hence, we could have neglected it.)

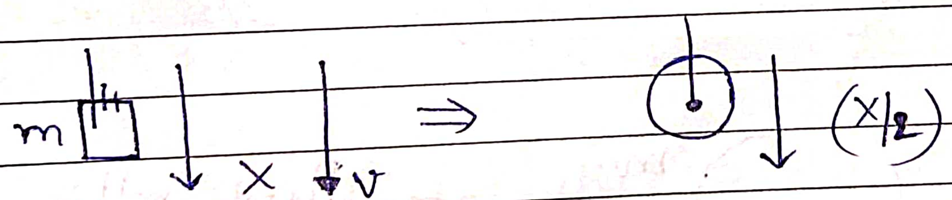
$$\Rightarrow \quad a = \left(\frac{-k}{m}\right)x \quad \Rightarrow \quad \omega^2 = k/m \Rightarrow \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

☆ If we have $Ax^2 + Bv^2 = \text{Const}$, then

$$\boxed{T = 2\pi\sqrt{\left|\frac{B}{A}\right|}}$$

Q) 

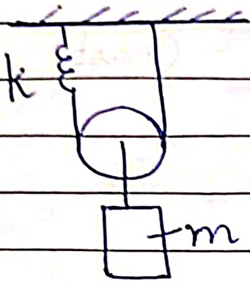
find time period

A) 

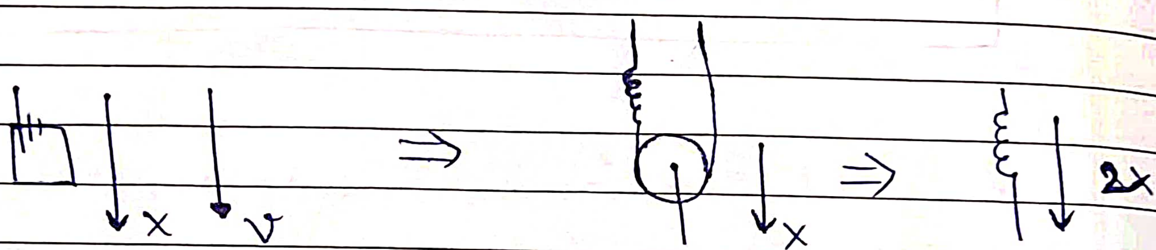
$$\frac{1}{2}mv^2 + \frac{1}{2}k\left(\frac{x}{2}\right)^2 = \text{Const.} \quad \Rightarrow \quad \boxed{T = 2\pi\sqrt{\frac{4m}{k}}}$$

$$\Rightarrow \left(\frac{m}{2}\right)v^2 + \left(\frac{k}{8}\right)x^2 = \text{Const.}$$

☆ $\boxed{\text{Jis obj. ka time period nikalna hai, usko 'x' se displace karo (i.e. uske CoM ko)}}$

Q) 

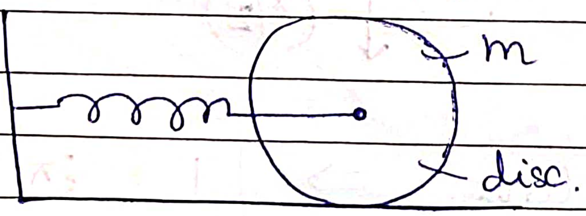
Find time period.

A) 

$\frac{1}{2}mv^2 + \frac{1}{2}k(2x)^2 = \text{Const.}$

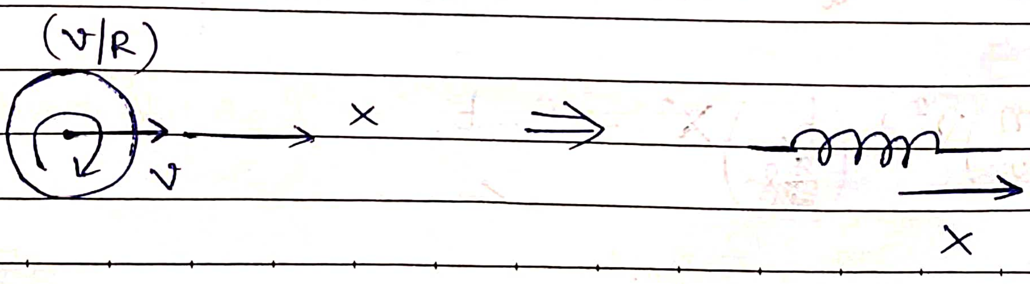
$\Rightarrow \left(\frac{m}{2}\right)v^2 + (2k)x^2 = \text{Const.}$

$\Rightarrow \boxed{T = \pi\sqrt{\frac{m}{k}}}$

Q) 

Pure rolling

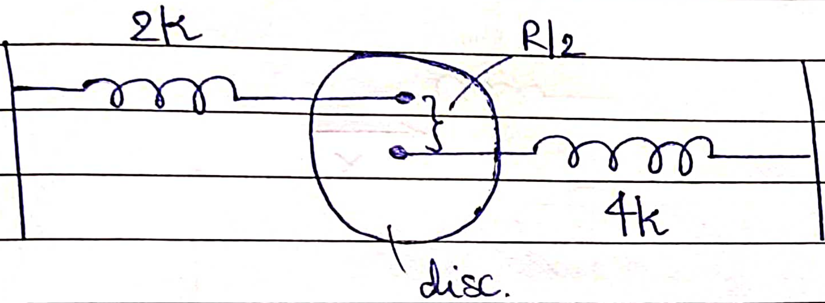
Find time period.

A) 

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 + \frac{1}{2}\cdot\frac{mR^2}{2}\cdot\left(\frac{v}{R}\right)^2 = \text{Const.}$$

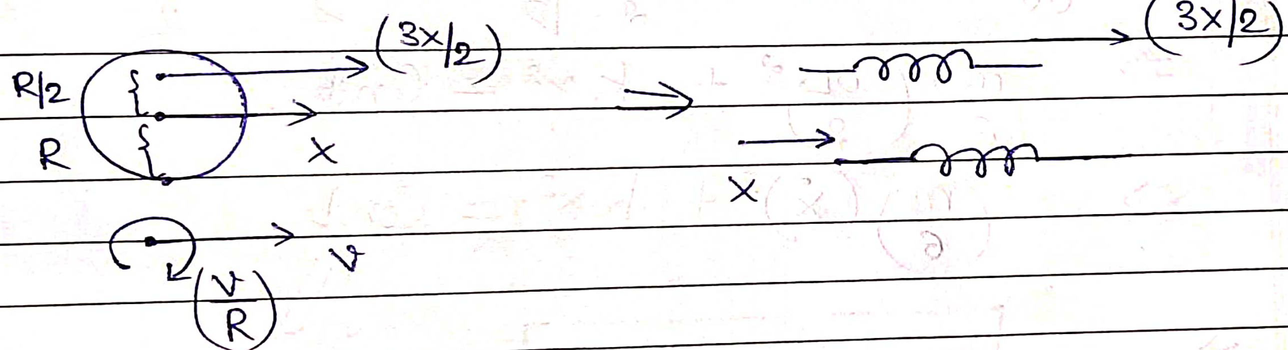
$$\Rightarrow \left(\frac{k}{2}\right)x^2 + \left(\frac{3m}{4}\right)v^2 = \text{Const.} \Rightarrow \boxed{T = (2\pi)\sqrt{\frac{3m}{2k}}}$$

Q)



Pure rolling
Small disp.
Find time period.

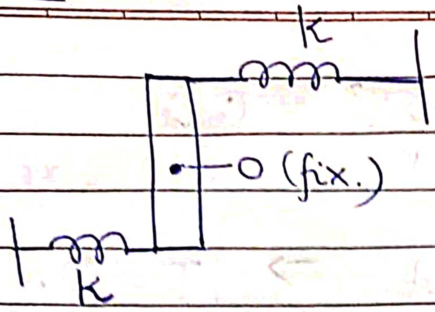
A)



$$\left[\frac{1}{2}(2k)\left(\frac{3x}{2}\right)^2 + \frac{1}{2}(4k)x^2\right] + \left[\frac{1}{2}mv^2 + \frac{1}{2}\cdot\frac{mR^2}{2}\cdot\left(\frac{v}{R}\right)^2\right] = \text{Const.}$$

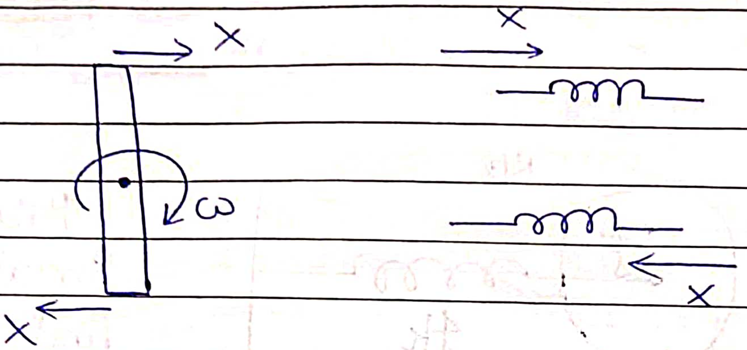
$$\Rightarrow \left(\frac{3m}{4}\right)v^2 + \left(\frac{17k}{4}\right)x^2 = \text{Const.} \Rightarrow \boxed{T = (2\pi)\sqrt{\frac{3m}{17k}}}$$

Q)



Small disp.
find time period.

A)



By Energy Consrv., $\frac{1}{2}\cdot\frac{mL^2}{12}\cdot\omega^2+\frac{1}{2}kx^2+\frac{1}{2}kx^2=\text{Const.}$

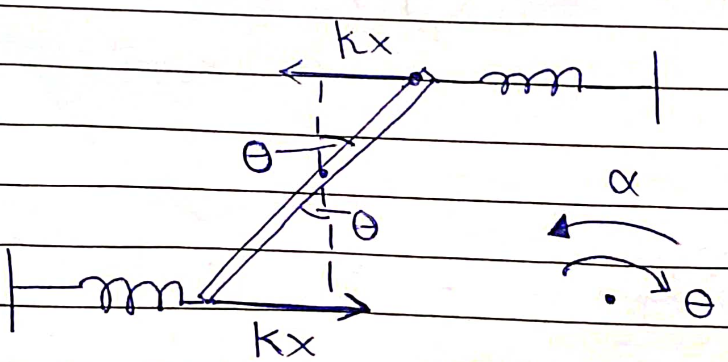
$\Rightarrow \quad \frac{m}{6}\cdot\left(\frac{\omega L}{2}\right)^2+kx^2=\text{Const}$

$\Rightarrow \quad \left(\frac{m}{6}\right)(\dot{x})^2+kx^2=\text{Const}.$

$\Updownarrow$

$\Rightarrow \quad \boxed{T=(2\pi)\sqrt{\frac{m}{6k}}}$ $\qquad \left(\frac{mL^2}{24}\right)\omega^2+\left(\frac{kL^2}{4}\right)\theta^2=\text{Const.}$

Alternative : Let rod rotate by θ.



$x=\left(\frac{\theta L}{2}\right)$

(abt O)

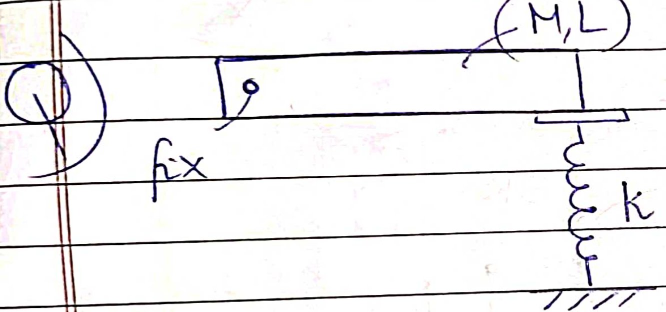
$$\tau = I\alpha \Rightarrow (2kx)(L/2) = \left(\frac{mL^2}{12}\right)(\alpha)$$

$$\Rightarrow \alpha = \left(\frac{12k}{mL}\right)x \Rightarrow \alpha = \left(\frac{6k}{m}\right)\theta$$

Angular S.H.M. $\Rightarrow \omega^2 = 6k/m$

$$\Rightarrow \boxed{T = 2\pi\sqrt{\frac{m}{6k}}}$$

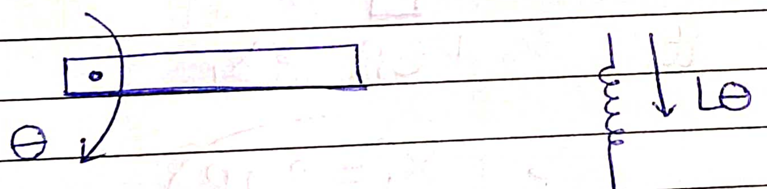
Q)



Obj. in eq.

Find time period if ~~not in eq.~~ obj displaced.

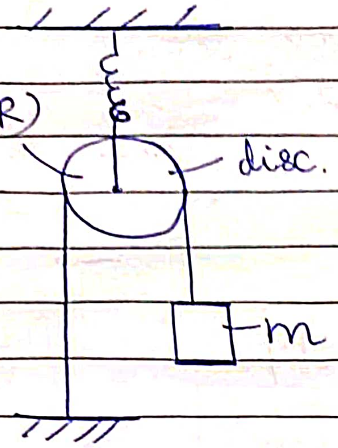
A)



$$\frac{1}{2}I\omega^2 + \frac{1}{2}k(L\theta)^2 = \text{Const.} \Rightarrow \left(\frac{mL^2}{6}\right)\omega^2 + \left(\frac{kL^2}{2}\right)\theta^2 = \text{Const.}$$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{m}{3k}}}$$

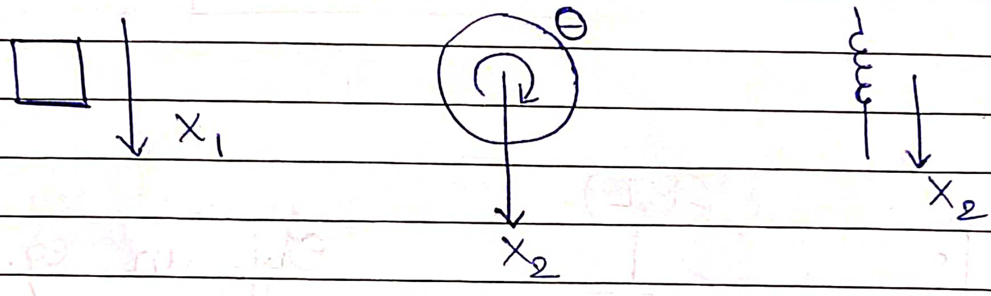
Q)



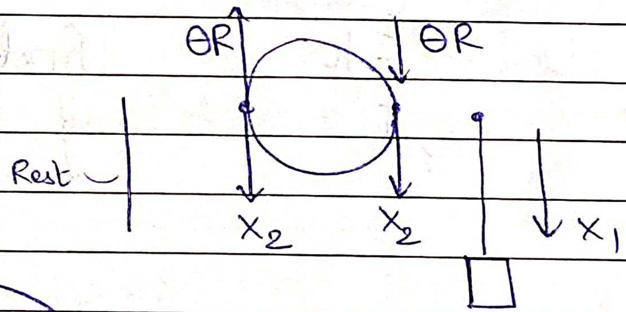
Find time period.

No slipping at any pt.

A)



Pure rolling :



$\Rightarrow$ $x_2 = \theta R$ & $x_2 + \theta R = x_1$

$\Rightarrow$ $x_1 = 2\theta R$

$\Rightarrow$ $(\dot{x}_2) = \omega R$

$\Rightarrow$ $(\dot{x}_1) = 2\omega R$

By Energy Consrv.,

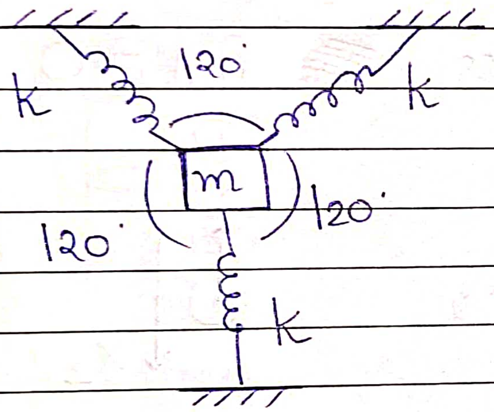
$$\frac{1}{2} m (\dot{x}_1)^2 + \frac{1}{2} k x_2^2 + \left[\frac{1}{2} M (\dot{x}_2)^2 + \frac{1}{2} \cdot \frac{MR^2}{2} \cdot \omega^2\right] = \text{Const.}$$

$$\Rightarrow \frac{1}{2}\cdot m\cdot 4\omega^2R^2 + \frac{1}{2}k\cdot\theta^2R^2 + \frac{1}{2}M\cdot\omega^2R^2 + \frac{MR^2}{4}\cdot\omega^2 = \text{Const.}$$

$$\Rightarrow (\omega^2)(R^2)\left[2m + \frac{3M}{4}\right] + (\theta^2)(R^2)\left[\frac{k}{2}\right] = \text{Const.}$$

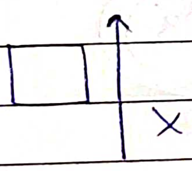
$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{8m+3M}{2k}}}$$

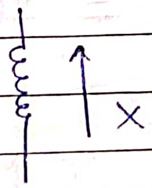
Q)

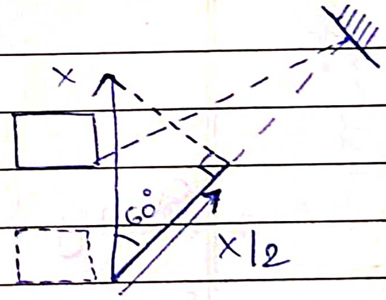


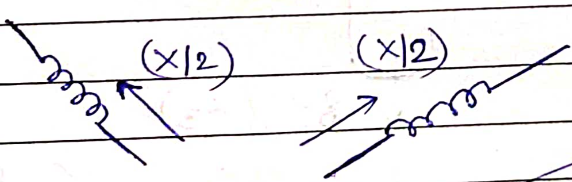
Find time period.

A)









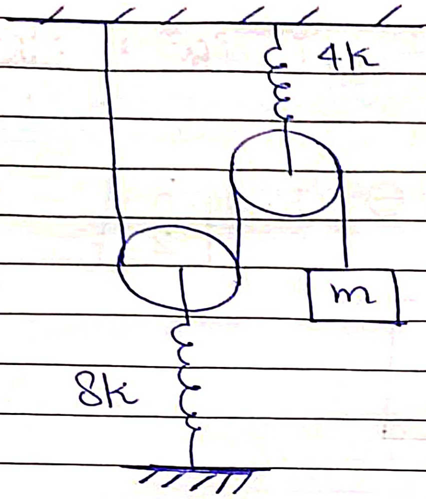
$$\boxed{T = (2\pi)\sqrt{\frac{2m}{3k}}}$$

By Energy Conserv.,

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 + \frac{1}{2}k\left(\frac{x}{2}\right)^2 + \frac{1}{2}k\left(\frac{x}{2}\right)^2 = \text{Const.}$$

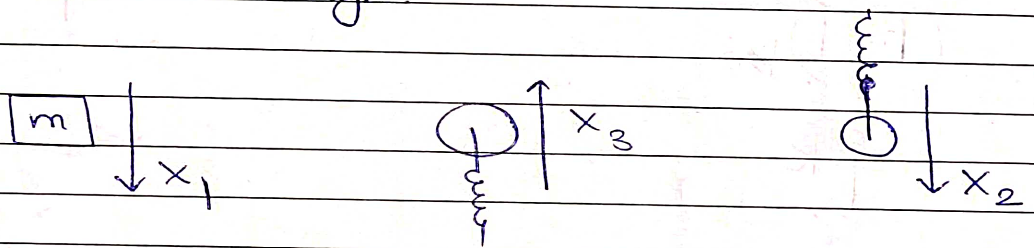
$$\Rightarrow (m/2)v^2 + (3k/4)x^2 = \text{Const.}$$

Q) ☆



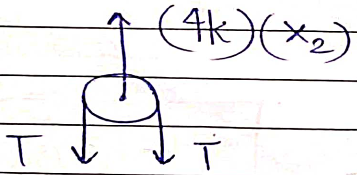
find time period.

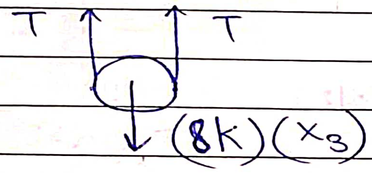
A) Since pulley's massless, ~~the~~ $F_{net} = 0$ on them always.



String Constraint : $x_1 = 2(x_2 + x_3)$

Eq. :





$2T = (4k)(x_2)$ $\qquad$ $2T = (8k)(x_3)$

$\Rightarrow$ $x_2 = 2x_3$

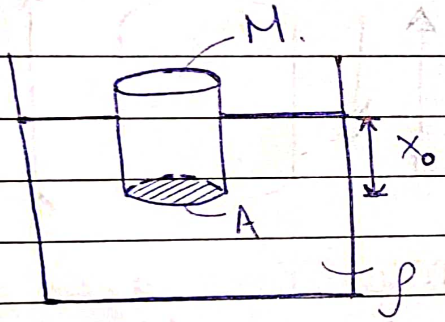
$\Rightarrow$ $x_1 = x$ (say), $x_2 = x/3$, $x_3 = x/6$

By Energy Consrv., $\frac{1}{2}mv^2 + \frac{1}{2}(8k)\left(\frac{x}{6}\right)^2 + \frac{1}{2}(4k)\left(\frac{x}{3}\right)^2 = \text{Const.}$

$\Rightarrow$ $(m/2)v^2 + (k/3)x^2 = \text{Const.}$

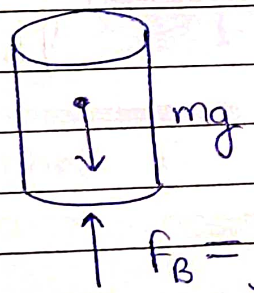
$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{3m}{2k}}}$$

Q)

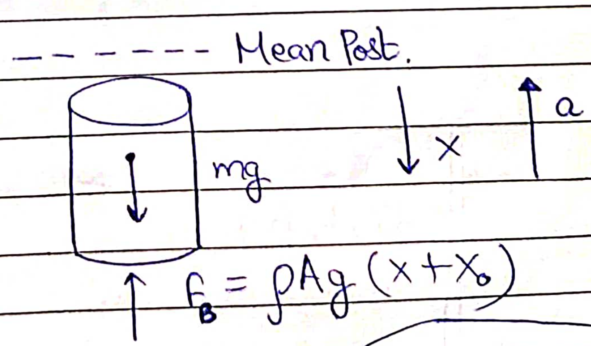


Find time period.

A)



$mg = \rho A g x_0$

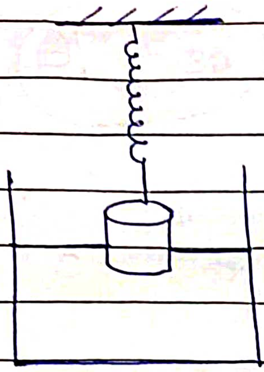


If obj. push down,

$F_{net} = \rho A g (x + x_0) - mg = \rho A g x \Rightarrow a = (\rho A g / m)x$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{m}{A\rho g}}}$$

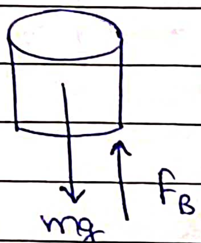
Q)



Mass $= M$
Area of Cross Section $= A$
Spring Const. $= K$.

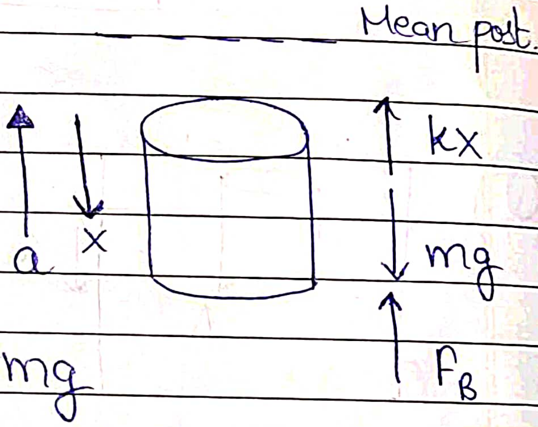
Find time period

A)



$F_B = A\rho g\, x_0 = mg$

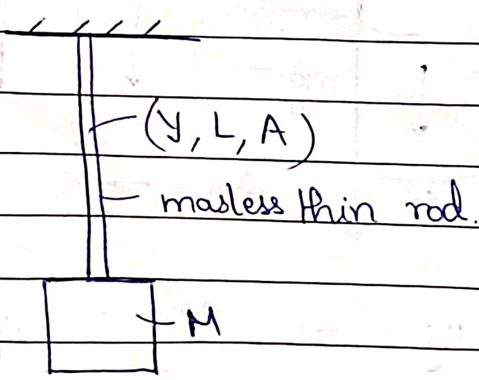
Now, push down the obj.



$F_{net} = kx + A\rho g(x + x_0) - mg$

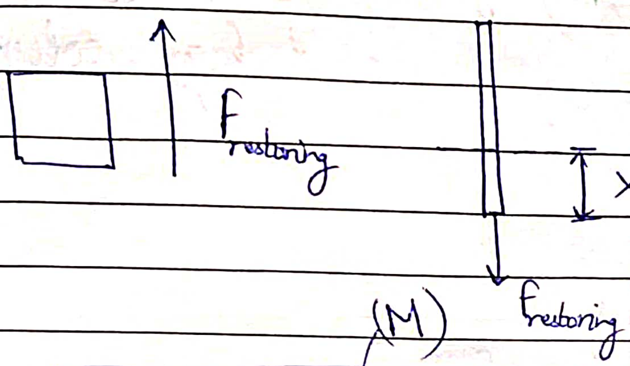
$\Rightarrow \quad a = \left(\dfrac{k + A\rho g}{m}\right) x \quad \Rightarrow \quad \boxed{T = (2\pi)\sqrt{\dfrac{m}{k + A\rho g}}}$

Q)



Find time period.

A) If obj. disp. down from mean posᵗ.



$$Y = \left(\frac{\text{Stress}}{\text{Strain}}\right)$$

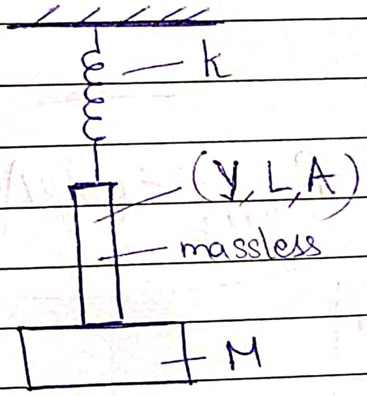
$$= \frac{f_{restoring}}{A(x/L)}$$

$$\Rightarrow \boxed{f = \left(\frac{AY}{L}\right)x}$$

$$\boxed{T = (2\pi)\sqrt{\frac{ML}{AY}}} \Leftarrow$$

(M)

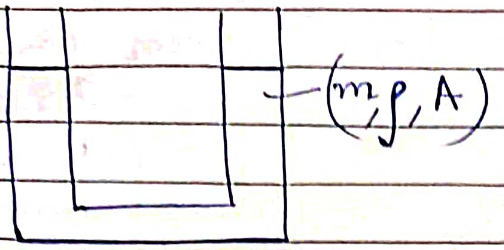
Q) find time period.



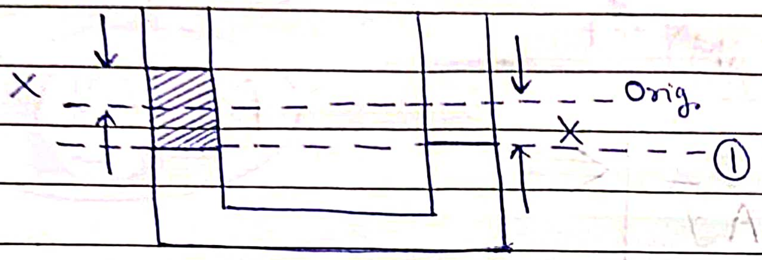
A) Series Connection $\Rightarrow$ $1/K_{eq} = 1/k + 1/(AY/L)$

$$\Rightarrow \frac{1}{K_{eq}} = \left(\frac{kL+AY}{AYk}\right)$$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{M(kL+AY)}{AYk}}}$$

Q) 

find time period of oscillation of liq. column

A) 

We find pressure at ①. P. diff. causes restoring force.

$$\Delta P = \left[P_0 + (2x)\rho g\right] - P_0$$

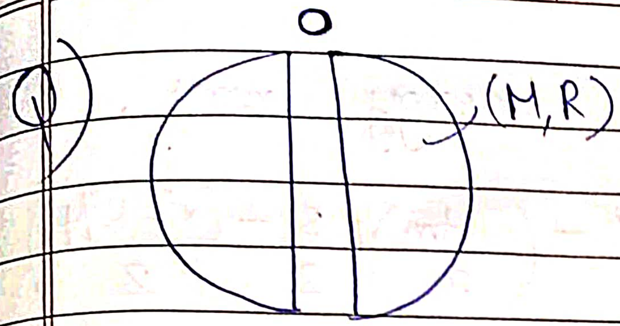
$$\Rightarrow \Delta P = 2x\rho g \Rightarrow F_{restoring} = (2\rho A g)x$$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{m}{2\rho A g}}}$$

☆ Since whole fluid performing SHM, we take mass 'm'.

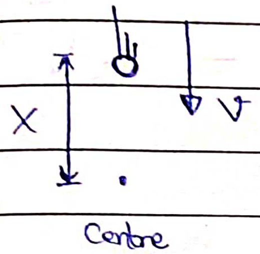
Otherwise, we will take mass of oscillating part only.

Q)



Find time period.

A)



$$f = \left(\frac{GMm}{R^2}\right)\left(\frac{x}{R}\right) = m\left(\frac{g}{R}\right)x$$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{R}{g}}}$$

Alternate: By Energy ~~Method~~ Consrv.

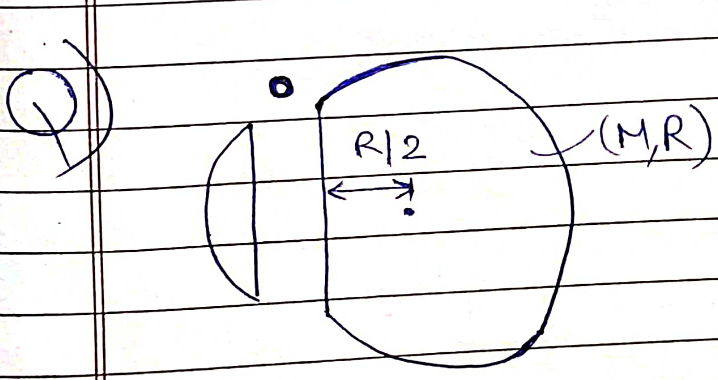
$$\cancel{\frac{GMm}{x}} + \frac{1}{2}mv^2 = \text{Const.}$$

$$\left(\frac{GMm}{R}\right)\left(\frac{-3}{2} + \frac{x^2}{2R^2}\right)$$

$$(m/2)v^2 + \left(mg/\cancel{R}\right)^{(2R)} x^2 = \text{Const.}$$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{R}{g}}}$$

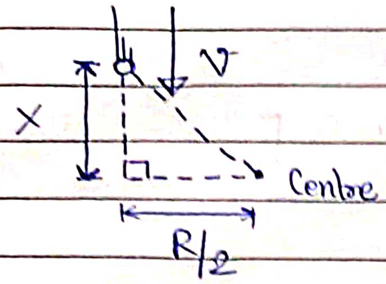
Q)



Find time period

372

Date: Page:

A) 

By Energy Conserv.,

$$\frac{1}{2}mv^2 - \left(\frac{GMm}{R}\right)\left(\frac{3}{2} - \frac{(x^2+R^2/4)}{2}\right) = \text{Const}$$

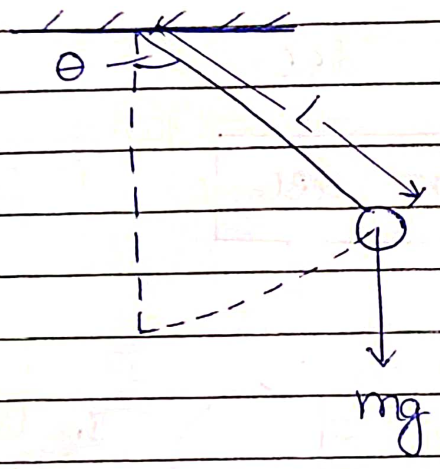
$$\Rightarrow (m/2)\,v^2 + (mg/2R)\,x^2 = \text{Const}.$$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{R}{g}}}$$

DATE ______
PAGE ______

Simple Pendulum

Pt. mass suspended from massless support with large length.



$$\tau = I\alpha$$

$$\Rightarrow (mg\sin\theta)L = (mL^2)\alpha$$

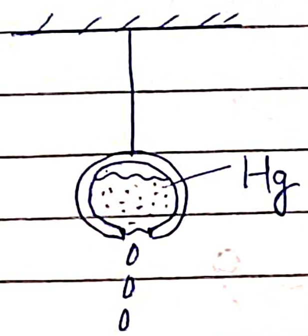
$$\Rightarrow \alpha = (g/L)\sin\theta$$

for small θ, $\sin\theta \approx \theta \Rightarrow \boxed{\alpha = (g/L)\theta}$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{L/g}}$$

L = Sep. b/w pt. of suspension & centre of gravity.

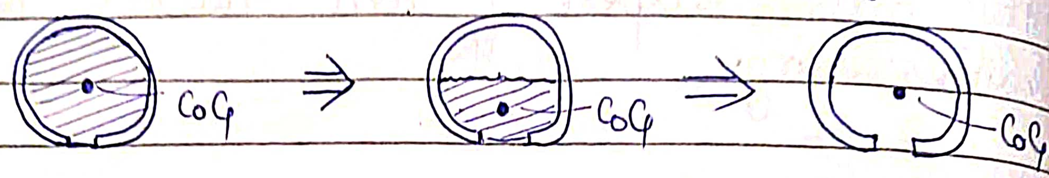
Q1) ☆



Comment on time period.

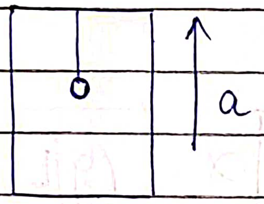
2

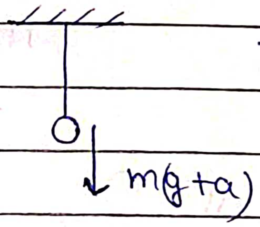
A) Slowly liq. drip ⇒ Dist. b/w support & CoG change



⇒ L first inc. then dec.

⇒ $\boxed{T \text{ first inc. then dec.}}$

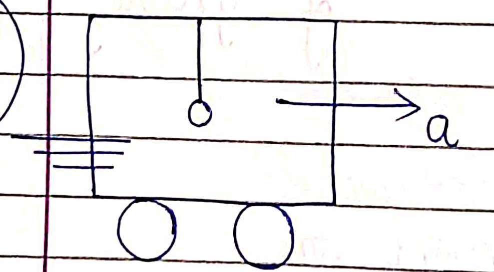
Q) 

A) 

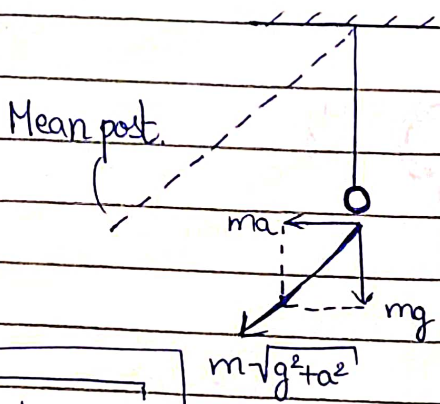
In lift's frame.

Find time period.

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{L}{(g+a)}}}$$

Q) 

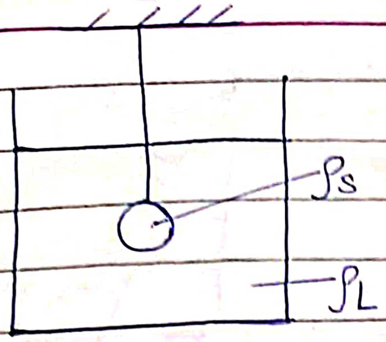
A) In car's frame



Find time period.

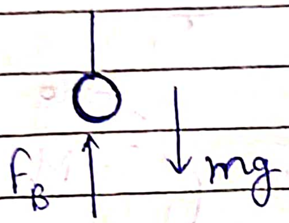
$g_{eff} = \sqrt{g^2+a^2}$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{L}{\sqrt{g^2+a^2}}}}$$

Q) 

$(\rho_S > \rho_L)$

Find time period.

A) 

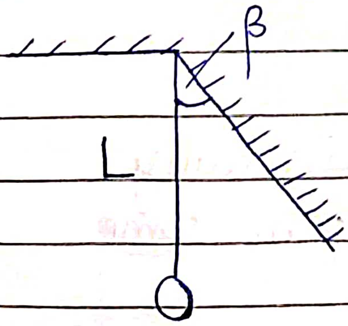
$$F_B = \rho_L V g = m(\rho_L/\rho_S)$$

$$F_{net} = mg - F_B$$

$$= (mg)(1 - \rho_L/\rho_S)$$

$$\Rightarrow \quad g_{eff} = g\left(1 - \frac{\rho_L}{\rho_S}\right)$$

$$\Rightarrow \quad \boxed{T = (2\pi)\sqrt{\frac{L}{g(1-\rho_L/\rho_S)}}}$$

Q) 

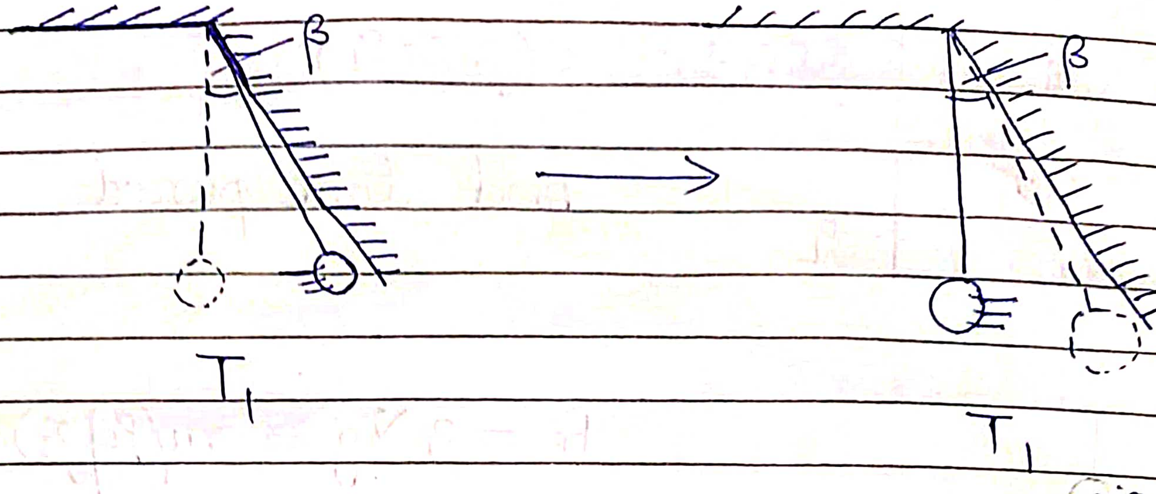
Angular amp $= \alpha$.
with $\alpha > \beta$.

Find time period.

4

DATE
PAGE

A)



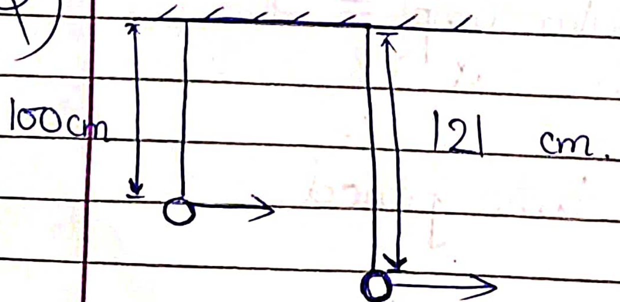
After this pendulum follow ~~full~~ half cycle of orig. SHM.

$$\Rightarrow \quad T = 2T_1 + \pi\sqrt{\frac{L}{g}}$$

Now, $\theta = \alpha \sin\left(t\sqrt{g/L}\right) \Rightarrow T_1 = \sqrt{\frac{L}{g}} \sin^{-1}\left(\frac{\beta}{\alpha}\right)$

$$\Rightarrow \quad \boxed{T = \left[\pi + 2\sin^{-1}(\beta/\alpha)\right]\sqrt{L/g}}$$

Q)



They initially start from same pt.

Find min. no of oscillation for them to acquire this config. again.

DATE
PAGE 5

A) $T_1 = (2\pi)\sqrt{\frac{100}{g}} \Rightarrow T_1 = (2\pi/\sqrt{g})(10)$

$T_2 = (2\pi)\sqrt{\frac{121}{g}} \Rightarrow T_2 = (2\pi/\sqrt{g})(11)$

They acquire same config. at $t = \left(\frac{2\pi}{\sqrt{g}}\right)(110)$

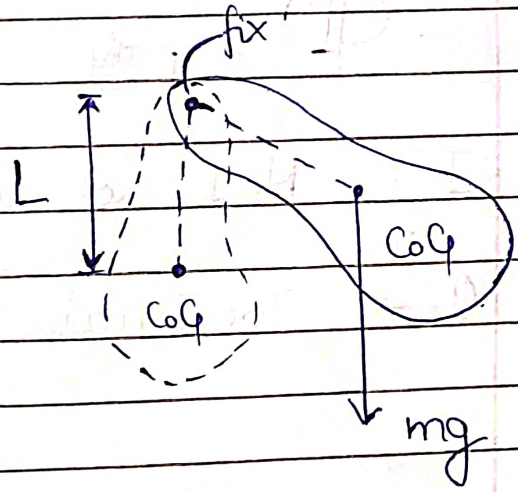
$\Rightarrow$ No. of oscillation by smaller pendulum = 11
" " " " larger " = 10

Compound Pendulum

$\tau = I\alpha$

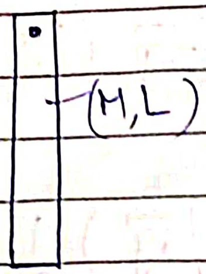
$\Rightarrow I\alpha = (mg\,r\theta)(L)$

$\Rightarrow \boxed{\alpha = \left(\frac{mgL}{I}\right)\theta}$



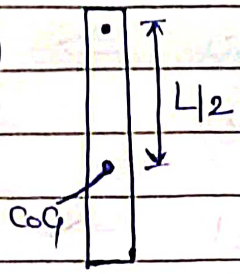
$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{I}{mgL}}}$ about AoR.

Q)



Find time period

A)

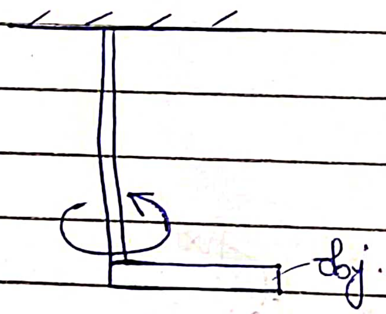


$$T = (2\pi)\sqrt{\frac{I}{mg(L/2)}}$$

$$= (2\pi)\sqrt{\frac{mL^2/3}{mg(L/2)}}$$

$$\Rightarrow \boxed{T = (2\pi)\sqrt{\frac{2L}{3g}}}$$

Torsional Pendulum



$$\boxed{T = (2\pi)\sqrt{\frac{I}{C}}}$$

I = MoI of obj. abt AoR.

C = Torsional const.

DATE ______
PAGE ______

Superposition of SHMs

Let SHM1 : $x_1 = a \sin(\omega t)$
& SHM2 : $x_2 = b \sin(\omega t + \phi)$

Net Motion : $x = x_1 + x_2 = a \sin(\omega t) + b \sin(\omega t + \phi)$

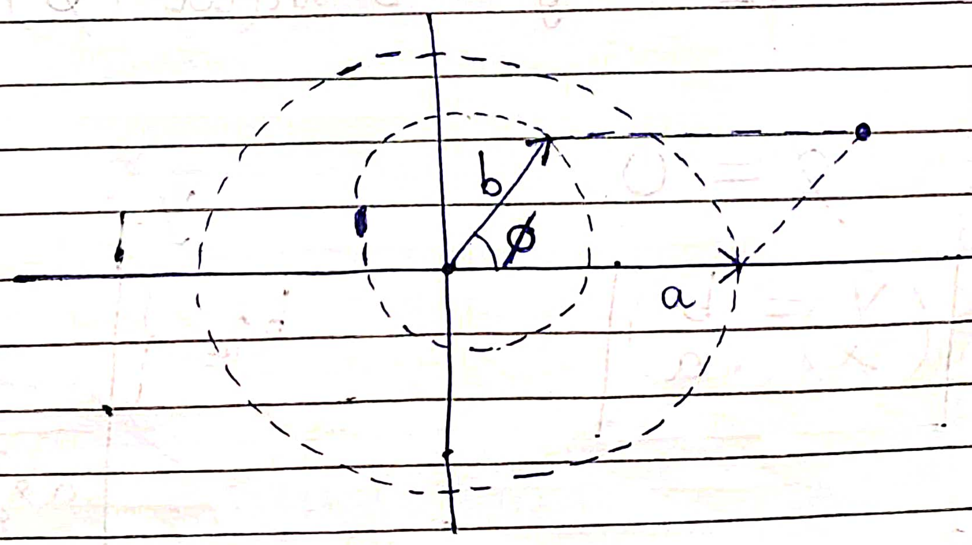
$$\Rightarrow x = [a + b\cos(\phi)] \sin(\omega t) + [b \sin(\phi)] \cos(\omega t)$$

$$\Rightarrow \boxed{x = A \sin(\omega t + \theta)}$$

where $A \cos(\theta) = a + b\cos(\phi)$
& $A \sin(\theta) = b \sin(\phi)$.

$$\Rightarrow \boxed{A = \sqrt{a^2 + b^2 + 2ab\cos(\phi)}} \quad ; \quad \boxed{\tan(\theta) = \left(\frac{b\sin(\phi)}{a + b\cos(\phi)}\right)}$$

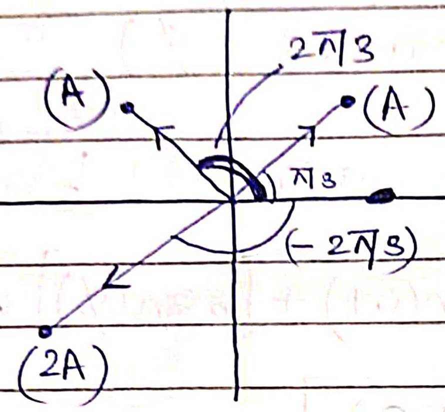
Phasar Diagram —



Q) $x_1 = A\sin(\omega t + \pi/3)$
$x_2 = A\sin(\omega t + 2\pi/3)$
$x_3 = (2A)\sin(\omega t + 4\pi/3)$

find eq^n of combined SHM

A)



$\vec{r}_1 = A\langle 1/2, \sqrt{3}/2\rangle$

$\vec{r}_2 = A\langle -1/2, \sqrt{3}/2\rangle$

$\vec{r}_3 = (2A)\langle -1/2, -\sqrt{3}/2\rangle$

$$(\vec{r}_1 + \vec{r}_2 + \vec{r}_3) = \vec{r} = (A)\langle -1, 0\rangle$$

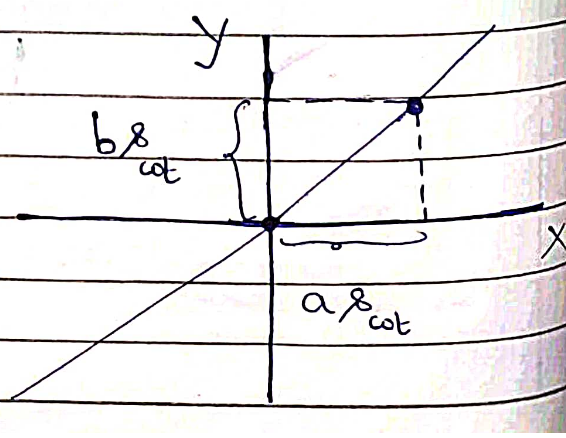
$$\boxed{x = A\sin(\omega t + \pi)}$$

Composition of 2 ⊥ SHMs

Let SHM1 : $x = a\sin(\omega t)$
& SHM2 : $y = b\sin(\omega t + \phi)$

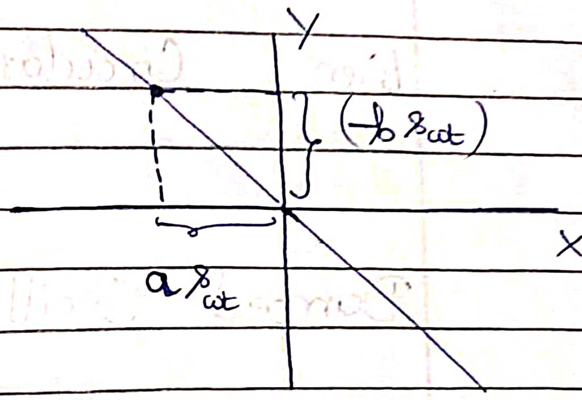
CL — $\phi = 0$

$$\Rightarrow \boxed{\left(\frac{y}{x}\right) = \left(\frac{b}{a}\right)}$$



$$\Rightarrow \boxed{A = \sqrt{a^2+b^2}}$$

C2 — $\phi = \pi$

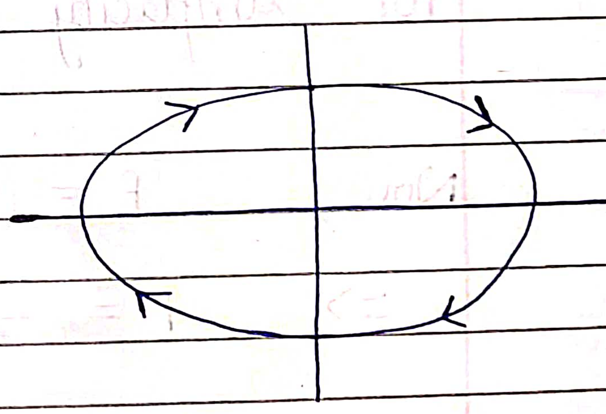


$$\Rightarrow \boxed{\left(\frac{y}{x}\right) = \left(\frac{-b}{a}\right)}$$

$$\Rightarrow \boxed{A = \sqrt{a^2+b^2}}$$

C3 — $\phi = \pi/2$

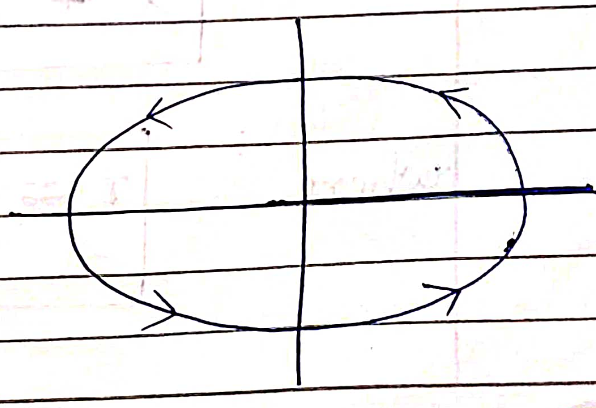
$$\Rightarrow \boxed{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1}$$



NOT an SHM

C4 — $\phi = 3\pi/2$

$$\Rightarrow \boxed{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1}$$



NOT an SHM

C5 — Else $\Rightarrow$ ~~Inclined~~ Oblique Ellipse.

In C3 & C4, if $a = b$

then Circular Motion.

Damped Oscillation

Damping force present, which acts opp. to dirxⁿ of motion.

For simplicity assume, $\boxed{\vec{F}_d = (-b)\vec{v}}$

damping const.

Now, $\vec{F} = (-k)\vec{x} + \vec{F}_d$

$\Rightarrow \vec{F} = -(k\vec{x} + b\vec{v})$

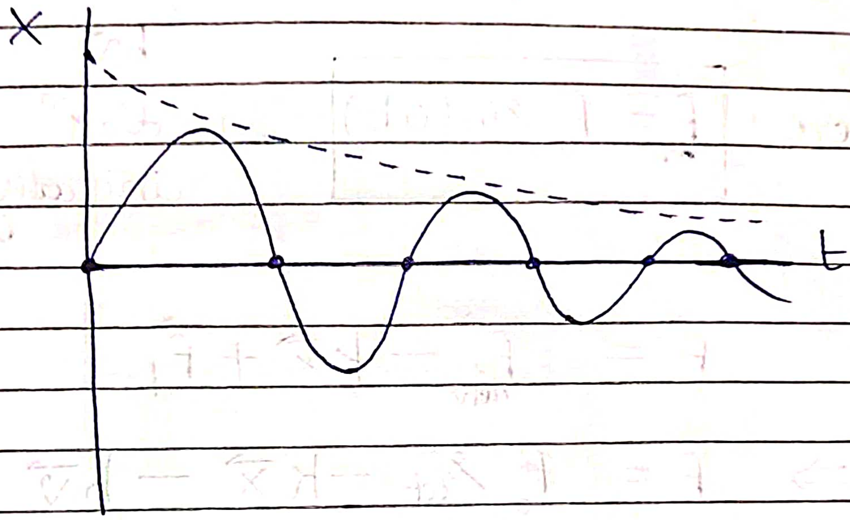
$\Rightarrow m\ddot{x} + b\dot{x} + kx = 0$

$\Rightarrow \boxed{\ddot{x} + 2r\dot{x} + \omega_0^2 x = 0}$

where $\boxed{r = \left(\frac{k}{2m}\right)}$ & $\boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

$\Rightarrow$ $\boxed{x = A_0 e^{-rt} \sin(\omega t)}$

where $\boxed{\omega^2 = (\omega_0^2 - r^2)}$



☆ $\boxed{\text{Time period does NOT change.}}$

Q) Amp. : $A_0 \longrightarrow A_0/3$ in = 2 min.
find amp. at $t = 6$ min.

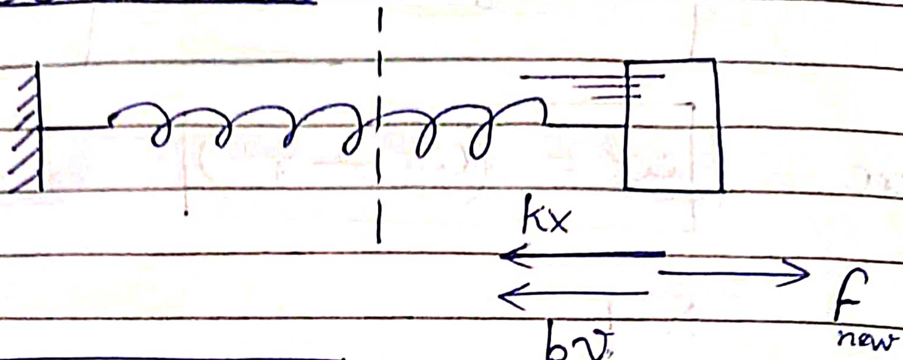
A) $A = A_0 e^{-rt}$ $\Rightarrow$ first Order (Kinetics)

$\Rightarrow$ $t_{1/3} = 2$

$A =$	A_0	$A_0/3$	$A_0/9$	$A_0/27$
$t =$	0	2	4	6

12

Forced Oscillation



where $\boxed{f_{new} = f_0 \sin(\omega t)}$ in dirx^n of motion initially.

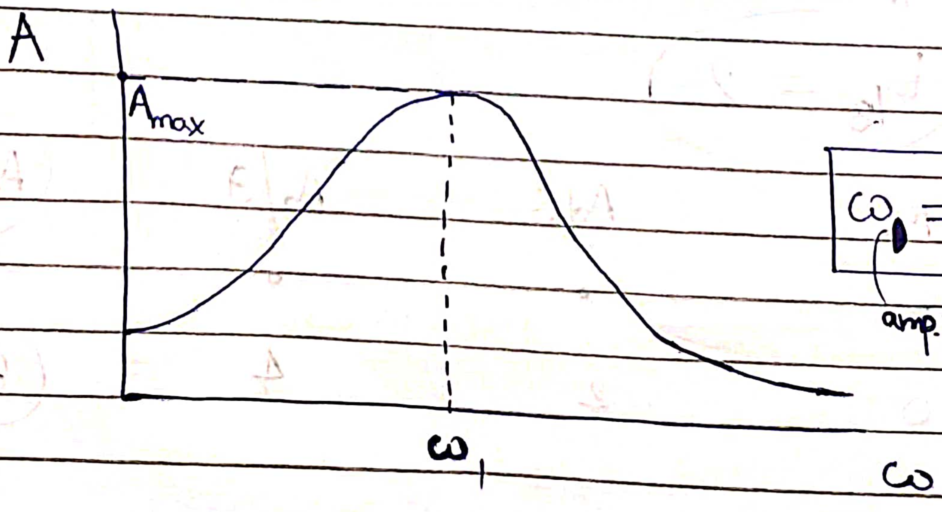
$$\vec{F} = \vec{f}_{new} - k\vec{x} + \vec{F}_d$$

$$\Rightarrow \vec{F} = f_0 s_{\omega t} - k\vec{x} - b\vec{v}$$

$$\Rightarrow m\ddot{x} = f_0 s_{\omega t} - kx - b\dot{x}$$

$$\Rightarrow \boxed{\ddot{x} + 2r\dot{x} + \omega_0^2 x = f_0 \sin(\omega t)}$$

where $\boxed{r = b/2m}$, $\boxed{\omega_0 = \sqrt{k/m}}$, $\boxed{f_0 = F_0/m}$.



$\boxed{\omega_1 = \sqrt{\omega_0^2 - 2r^2}}$

amp. res.

$$\boxed{A_{max.} \propto 1/r}$$

Now,
$$\boxed{A = \frac{f_0}{\sqrt{(\omega^2 - \omega_0^2)^2 + (b\omega/m)^2}}}$$

Resonance

1) Amplitude Res. — freq. of periodic force at which amp. is max.

$$\boxed{\omega_{amp.res.} = \sqrt{\omega_0^2 - 2r^2}}$$

2) Velocity Res. — freq. of periodic force at which vel. (or energy) is max.

$$\boxed{\omega_{vel.\ res.} = \omega_0}$$

DATE

PAGE

Q) 2 particles oscillating in SHM of const. amp. They always meet at $x = \left(\frac{A}{2}\right)$, moving in opp. dir$x^n$. Find phase diff.

A) Let eq^{n}s be $x_1 = A \sin(\omega_1 t + \phi_1)$ & $x_2 = A \sin(\omega_2 t + \phi_2)$

Always meet at $x = A/2 \Rightarrow \omega_1 = \omega_2$

Let at $t = 0$, both be at $x = A/2$.

$\Rightarrow A \sin\phi_1 = A \sin\phi_2 = A/2$

$\Rightarrow \sin\phi = 1/2$ — ϕ_1, ϕ_2

$\Rightarrow \phi_1 = \pi/6$ & $\phi_2 = 5\pi/6 \Rightarrow$ $\boxed{\text{Phase diff.} = 2\pi/3}$

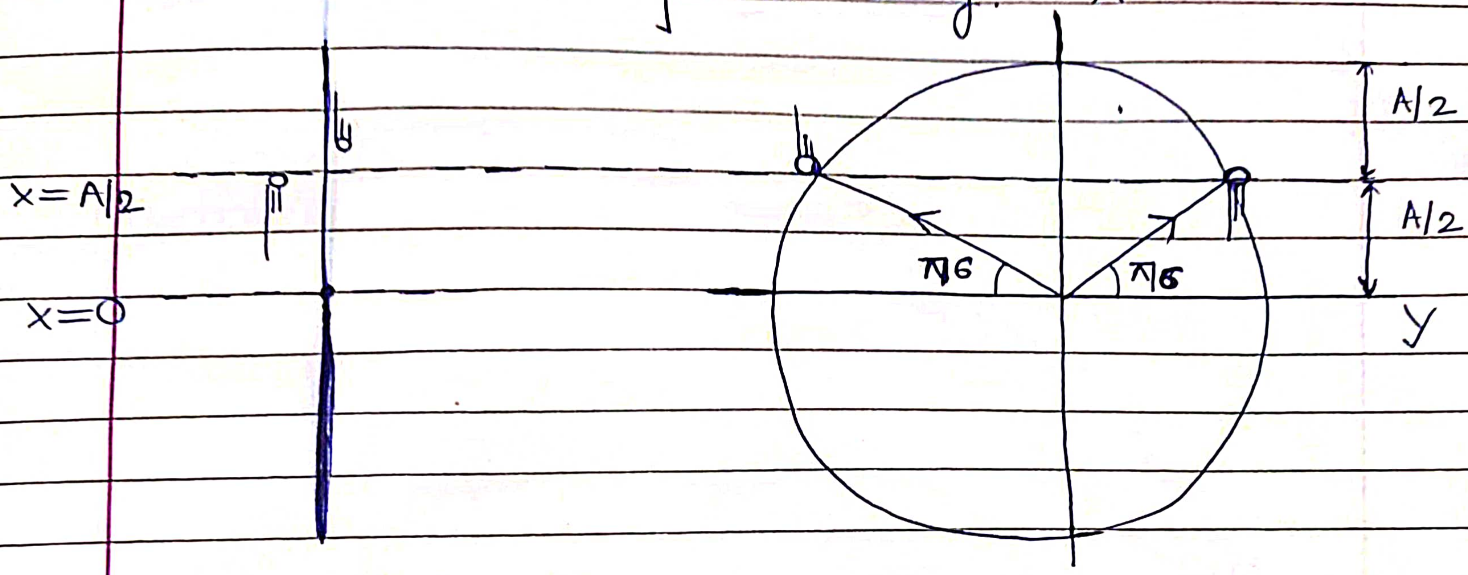
Alternate : Let eq^{n}s be $x_1 = A \sin\omega t$
& $x_2 = A \sin(\omega t + \phi)$

At $t = t_1 \Rightarrow A/2 = A \sin\omega t_1 \Rightarrow \sin\omega t_1 = 1/2$

Also, $A/2 = A \sin(\omega t_1 + \phi) \Rightarrow \sin(\omega t_1 + \phi) = 1/2$

$\Rightarrow \boxed{\phi = 2\pi/3}$

Alternate: Use phasar diag.



Consider disp. of particle as proj. of rotating vectors on coor. axis.

$$\Rightarrow \boxed{\text{Phase Diff.} = 2\pi/3}$$